SOLUTIONS OF PROBLEMS.

ALGEBRA.

462. Proposed by H. S. UHLER, Yale University.

Show how to transform A into S, where these symbols denote the equivalent formulæ for the general case of Calculus Problem No. 363, pages 52 and 54 in the February, 1916, MONTHLY:

$$A \equiv 4\pi R^2 - 4nR^2 \sin^{-1}\left(\frac{R\sin\pi/n}{\sqrt{R^2-a^2}}\right) + 2anR\sin^{-1}\left[\frac{2a(\tan\pi/n)(\sqrt{R^2-a^2\sec^2\pi/n})}{R^2-a^2}\right],$$

$$S \equiv 4nR\left\{a\sin^{-1}\left(\tan\frac{\pi}{n}\cdot\frac{a}{\sqrt{R^2-a^2}}\right) - R\sin^{-1}\left[\tfrac{1}{2}\sin\frac{2\pi}{n}\cdot\frac{R-\sqrt{R^2-a^2\sec^2\pi/n}}{\sqrt{R^2-a^2}}\right]\right\}.$$

SOLUTION BY GEO. W. HARTWELL, Hamline University.

$$A \equiv 4nR\left\{R\left[\frac{\pi}{n} - \sin^{-1}\left(\frac{R\sin\pi/n}{\sqrt{R^2-a^2}}\right)\right] + \tfrac{1}{2}a\sin^{-1}\left[\frac{2a(\tan\pi/n)(\sqrt{R^2-a^2\sec^2\pi/n})}{R^2-a^2}\right]\right\}$$

$$\equiv 4nR(a\gamma + R\beta).$$

Let

$$\sin^{-1}\left(\frac{R\sin\pi/n}{\sqrt{R^2-a^2}}\right) = \alpha \quad \text{and} \quad \frac{\pi}{n} - \alpha = \beta.$$

Then,

$$\sin\beta = \sin\frac{\pi}{n}\cos\alpha - \cos\frac{\pi}{n}\sin\alpha.$$

$$\cos\alpha = \sqrt{1 - \frac{R^2\sin^2\pi/n}{R^2-a^2}} = \sqrt{\frac{R^2\cos^2\pi/n - a^2}{R^2-a^2}} = \frac{\cos\pi/n\sqrt{R^2-a^2\sec^2\pi/n}}{\sqrt{R^2-a^2}}.$$

Hence,

$$\sin\beta = \sin\frac{\pi}{n}\cos\frac{\pi}{n}\left[\frac{\sqrt{R^2-a^2\sec^2\pi/n} - R}{\sqrt{R^2-a^2}}\right],$$

or

$$\beta = -\sin^{-1}\left[\tfrac{1}{2}\sin\frac{2\pi}{n}\cdot\frac{R-\sqrt{R^2-a^2\sec^2\pi/n}}{\sqrt{R^2-a^2}}\right].$$

Let

$$\sin^{-1}\frac{2a(\tan\pi/n)(\sqrt{R^2-a^2\sec^2\pi/n})}{R^2-a^2} = \gamma.$$

Then,

$$\sin\tfrac{1}{2}\gamma = \sqrt{\tfrac{1}{2}(1-\cos\gamma)} = \sqrt{\frac{1}{2}\left(1 - \sqrt{1 - \frac{4a^2\tan^2\pi/n\,(R^2-a^2\sec^2\pi/n)}{(R^2-a^2)^2}}\right)}$$

$$= \sqrt{\frac{1}{2}\left(1 - \frac{\sqrt{R^4 - 2a^2R^2(1+2\tan^2\pi/n) + a^4(1+4\sec^2\pi/n\tan^2\pi/n)}}{R^2-a^2}\right)}$$

$$= \sqrt{\frac{1}{2}\left(1 - \frac{R^2 - a^2(1+2\tan^2\pi/n)}{R^2-a^2}\right)} = \frac{a\tan\pi/n}{\sqrt{R^2-a^2}};$$

hence,

$$\tfrac{1}{2}\gamma = \sin^{-1}\frac{a\tan\pi/n}{\sqrt{R^2-a^2}}.$$

Making these substitutions, we have $A \equiv S$.

Also solved by the PROPOSER.